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High-frequency sound waves in ideal gases with internal dissipation

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High-frequency plane sound waves in ideal gases with internal dissipation are discussed. Particular applications to dissociating diatomic gases and gases displaying vibrational relaxation are considered. A criterion in the form of an inequality is derived for the validity of the high-frequency approximation and an asymptotic analysis is developed.

1. Introduction

Equations of state with one rate-dependent state variable arise in the study of gases subject to chemical dissociation or vibrational relaxation. In the former case the possible effects of diffusion are normally neglected so that the purely chemical phenomenon is treated in isolation. Comprehensive review articles on this field and its applications have been written by Li (1961) and Lick (1967). The latter of these articles deals mainly with the subject of one-dimensional wave propagation governed by linearized equations.

Recently Coleman & Gurtin (1967*a*) have laid down a thermodynamics of elastic materials with internal state variables. This theory includes as a special case that for an ideal gas, and, with this as background, they have investigated the propagation of plane discontinuities of order two or more (Coleman & Gurtin 1967*b*). In particular they give detailed consideration to the behaviour of the strength of the discontinuity along the 'leading characteristic'. A similar though less thorough treatment has been given by Rarity (1967). In both these analyses the predictions of the linear theory are shown to be in error.

On the basis of a 'relatively undistorted' wave approximation, alluded to and applied by Varley & Cumberbatch (1966), we shall investigate plane high-frequency waves in dissipative gases. The meaning of this terminology becomes clear when the same are considered in an ideal monatomic gas. In this case a one-parameter family of solutions exists such that the values of the field variables remain constant on the characteristics or 'wavelets' which in turn are described by straight lines. When a dissipative mechanism is introduced, then such is not the case. However, the curvature of the characteristics may under certain circumstances be slight.

The term undistorted wave is introduced by Courant & Hilbert (1962) in their discussion of dispersion in plane waves governed by linear hyperbolic equations. In cases where no dissipative terms are present a one-parameter family of solutions in terms of the phases or characteristics $(x-at)$, where a is constant, may be found. When a dissipative term is introduced, then one-parameter families of solutions may still be obtained but they are of exponential type while the phase speeds must be less than a. Thus the different phases of a wave packet travel with different speeds leading to dispersion.

Here it is shown that the linear theory is incorrect for high-frequency sound waves in gases with internal dissipation. In fact the linear theory predicts that high-frequency waves propagate undistorted with the 'frozen' wave speed while predicting the correct form of attentuation. A correction to the basic approximation is obtained through a formal asymptotic analysis and the phenomenon of dispersion is displayed. Similar techniques have been employed by Lick (1967) but there the basic approximation is the linearization, which is more suited to an analysis of small-amplitude low-frequency waves.

2. Field equations and relatively undistorted waves

The equations governing one-dimensional time-dependent motions in an ideal gas with internal dissipation have the form

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial t}+\mathbf{B}\frac{\partial \mathbf{u}}{\partial x}+\mathbf{C} = 0, \tag{2.1}$$

where

$$\mathbf{u}^T = [u,\rho,p,\alpha],$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \dfrac{\partial H}{\partial \rho} & \left(\dfrac{\partial H}{\partial p}-\dfrac{1}{\rho}\right) & \dfrac{\partial H}{\partial \alpha} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} u & 0 & \rho^{-1} & 0 \\ \rho & u & 0 & 0 \\ 0 & u\dfrac{\partial H}{\partial \rho} & u\left(\dfrac{\partial H}{\partial p}-\dfrac{1}{\rho}\right) & u\dfrac{\partial H}{\partial \alpha} \\ 0 & 0 & 0 & u \end{bmatrix},$$

and

$$\mathbf{C} = [0,0,0,f(p,\rho,\alpha)].$$

Here u is the material velocity, p the pressure, ρ the density and α an internal state variable which may either represent the degree of dissociation in a diatomic gas or be some measure of vibrational energy in the same. $H = H(p,\rho,\alpha)$ is the specific enthalpy.

We will be concerned here with the propagation of waves into an equilibrium state. An equilibrium state is defined at a point (p_0, v_0, α_0) in state space as

$$f(p_0, v_0, \alpha_0) = 0, \tag{2.2}$$

where $v = 1/\rho$. Such a state is said to be (locally) asymptotically stable at constant pressure and volume if the solution $\alpha(t)$ of

$$\frac{D\alpha}{Dt} = -f(p_0, v_0, \alpha): \quad \alpha(0) = \alpha^* \tag{2.3}$$

exists for all $t > 0$ and $\alpha(t) \to \alpha_0$ as $t \to \infty$ for all α^* such that $|\alpha^*-\alpha_0| < \zeta$, $\zeta > 0$. This parallels the definition given by Coleman & Gurtin (1967*a*), and by further paraphrasing of the arguments there it may be shown from asymptotic stability that

$$\left(\frac{\partial H(p_0, v_0, \alpha)}{\partial \alpha} - \theta\frac{\partial \eta(p_0, v_0, \alpha)}{\partial \alpha}\right)\bigg|_{\alpha=\alpha_0} = 0, \tag{2.4}$$

where $\eta(p_0, v_0, \alpha)$ is the entropy and θ the temperature. The quantity

$$-\{(\partial H/\partial \alpha) - \theta(\partial \eta/\partial \alpha)\}$$

is referred to as the affinity of reaction (Prigogine & Defay 1954) and in the case of chemical reactions (2.4) is often inferred as a statement of equilibrium from chemical kinetics or an Onsager principle by relating $f(p, \rho, \alpha)$ to the 'affinity' (De Groot 1951, Prigogine 1963).

In the case of a binary dissociated ideal gas the enthalpy has the form (Prigogine & Defay 1954)

$$H = \sum_{i=1}^{2} \frac{c_i}{M_i} h_i, \tag{2.5}$$

where h_i is the partial molar enthalpy given by

$$h_i = \epsilon(\theta) + R\theta, \tag{2.6}$$

and c_i is the concentration by mass ρ_i/ρ of species i, M_i is its molecular weight and R the Universal gas constant. Also Dalton's law of partial pressures

$$p = \rho R\theta \sum_{i=1}^{2} \frac{c_i}{M_i} \tag{2.7}$$

is obeyed. The concentrations satisfy the relation

$$\sum_{i=1}^{2} c_i = 1,$$

and c_1 is chosen as α, the degree of dissociation. The following quantities are defined:

$$\left.\begin{aligned}
&h_{\theta,p} = \left(\frac{\partial H}{\partial \alpha}\right)_{\theta,p}, \quad \text{heat of reaction,}\\
&\left.\begin{aligned} C_{v,\alpha} &= \frac{\partial}{\partial \theta}(H - p/\rho)_{v,\alpha},\\ C_{p,\alpha} &= \left(\frac{\partial H}{\partial \theta}\right)_{p,\alpha}, \end{aligned}\right\} \quad \text{specific heat capacities,}\\
&C_{p,\alpha} - C_{v,\alpha} = (1+\alpha)R_2, \quad R_2 = R/M_2,\\
&\left(\frac{\partial H}{\partial \rho}\right)_{p,\alpha} = -\frac{p}{\rho^2}\frac{C_{p,\alpha}}{(1+\alpha)R_2}, \quad \left(\frac{\partial H}{\partial p} - \frac{1}{\rho}\right)_{\rho,\alpha} = \frac{C_{v,\alpha}}{\rho(1+\alpha)R_2},\\
&\gamma = C_{p,\alpha}/C_{v,\alpha}, \quad a^2 = -\left(\frac{\partial H}{\partial \rho}\right)_{p,\alpha} \Big/ \left(\frac{\partial H}{\partial p} - \frac{1}{\rho}\right) = \frac{\gamma p}{\rho}.
\end{aligned}\right\} \tag{2.8}$$

It is assumed that $C_{p,\alpha}$ and $C_{v,\alpha}$ are functions of α only and have polynomial forms for small changes in α. It follows from (2.8) that

$$h_{\theta,p} = h_{\theta_0,p_0} + \frac{\partial C_{p,\alpha}}{\partial \alpha}\theta. \tag{2.9}$$

If the further assumption that h_{θ_0,p_0} is constant is made then the enthalpy has the form

$$H = pC_{p,\alpha}/\rho(1+\alpha)R_2 + \alpha d, \quad d = h_{\theta_0,p_0}, \tag{2.10}$$

or equivalently,

$$H = a^2/(\gamma - 1) + \alpha d. \tag{2.11}$$

A relation of the form (2.11) with $C_{p,\alpha} = \frac{1}{2}(7+3\alpha)R_2$ has been derived from kinetic-theory arguments (Li 1961).

In the case of vibrational relaxation the enthalpy is taken to have the form

$$H = \epsilon_T(\theta)+\epsilon_{\text{vib}}(\alpha)+R\theta/M \tag{2.12}$$

(i.e. it is ideal and θ and α are the 'translational' and vibrational temperatures respectively), while

$$p = \rho R\theta/M. \tag{2.13}$$

Through the stability condition (2.4), it may be shown that (2.12) implies, at equilibrium,

$$\alpha_0 = \hat{\alpha}_0(\theta). \tag{2.14}$$

If it is further assumed that the specific heats are independent of the temperatures θ and α, while $h_{\theta,p}$ is also independent of θ and α, then the enthalpy has the form

$$H = pC_p/\rho R_2+\alpha d = \frac{a^2}{\gamma-1}+\alpha d. \tag{2.15}$$

The expression (2.15) is again similar to one derived from kinetic-theory arguments (Li 1961). The relations (2.8) still hold in this case with the exception that $\alpha = 0$ where it appears explicitly.

In each case considered $f(p,\rho,\alpha)$ is left arbitrary.

A 'wavelet' is defined by a curve $\phi = \phi(x,t)$, and, on the assumption that $\partial\phi/\partial t \neq 0$, then an equivalent description is $t = T(x,\phi)$. Under the transformation of co-ordinates (x,t) to (x,ϕ) any vector-valued function $\mathbf{u}$ transforms as

$$\mathbf{u}(x,T(x,\phi)) = \mathbf{U}(x,\phi). \tag{2.16}$$

A relatively undistorted wave is defined by the relation

$$\left\|\frac{\partial\mathbf{U}}{\partial x}\right\| \ll \left\|\frac{\partial\mathbf{u}}{\partial x}\right\|, \tag{2.17}$$

where $\|.\|$ denotes the Euclidean norm of a vector, and since

$$\frac{\partial\mathbf{U}}{\partial x} = \frac{\partial\mathbf{u}}{\partial x}+\frac{\partial T}{\partial x}.\frac{\partial\mathbf{u}}{\partial t}, \tag{2.18}$$

then (2.17) implies that

$$\frac{\partial\mathbf{u}}{\partial x} \simeq -\frac{\partial T}{\partial x}.\frac{\partial\mathbf{u}}{\partial t}. \tag{2.19}$$

Equation (2.19) holds exactly at an acceleration wave-front propagating into an undisturbed region in thermodynamic equilibrium and also on all other 'wavelets' in a non-dissipative gas as then $\partial\mathbf{U}/\partial x = 0$, i.e. $\mathbf{u}(x,T(x,\phi)) = \mathbf{U}(\phi)$ is a solution.

We may write for (2.1)

$$\left(\mathbf{B}\frac{\partial T}{\partial x}-\mathbf{A}\right)\frac{\partial\mathbf{u}}{\partial t} = \mathbf{B}\frac{\partial\mathbf{U}}{\partial x}+\mathbf{C}, \tag{2.20}$$

so that (2.17) and (2.20) are compatible if

$$\frac{\|\mathbf{C}\|}{\|\mathbf{B}\,\partial\mathbf{U}/\partial x\|} = O(1), \tag{2.21}$$

while
$$\det(\mathbf{B} - W^{-1}\mathbf{A}) = 0, \quad W = \frac{\partial T}{\partial x}. \tag{2.22}$$

Thus $(\partial T/\partial x)^{-1}$ is an eigenvalue of $\mathbf{BA}^{-1}$ and the 'wavelets' are the characteristics of (2.1). As a consequence of (2.22) $\mathbf{u}$ must satisfy the compatibility condition

$$\boldsymbol{l}\left(\mathbf{B}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{C}\right) = 0, \tag{2.23}$$

where $\boldsymbol{l}$ is the left eigenvector associated with the eigenvalue W^{-1}.

The conditions (2.17) and (2.21) may be satisfied at a near equilibrium state. Then to a first approximation

$$\left(\mathbf{B}\frac{\partial T}{\partial x} - \mathbf{A}\right)\frac{\partial \mathbf{U}}{\partial \phi} = 0, \tag{2.24}$$

and equations (2.24), (2.22), (2.23) and (2.10) or (2.15) then determine $\mathbf{U}(x, \phi)$ and $T(x, \phi)$ subject to suitable boundary conditions.

3. First approximation

The solution of (2.22) yields the eigenvalue

$$\begin{aligned}\left(\frac{\partial T}{\partial x}\right)^{-1} &= W^{-1} = u + \sqrt{\left\{-\frac{\partial H}{\partial \rho}\Big/\left(\frac{\partial H}{\partial p} - \frac{1}{\rho}\right)\right\}} \\ &= u + a,\end{aligned} \tag{3.1}$$

where a is the local sound speed defined in (2.8). The left eigenvector associated with the eigenvalue (3.1) is

$$\boldsymbol{l} = \left[\rho\frac{\partial H}{\partial \rho}, \quad a\frac{\partial H}{\partial \rho}, \quad -a, \quad a\frac{\partial H}{\partial \alpha}\right]. \tag{3.2}$$

When (3.1) is substituted in (2.24) and use is made of the relation $a^2 = \gamma p/\rho$, the resulting equations may be integrated to give the solution appropriate to a plane wave propagating into a region in thermomechanical equilibrium,

$$\alpha = \alpha_0, \quad p = \rho^\gamma, \quad u = \frac{2\gamma^{\frac{1}{2}}}{\gamma - 1}\{\rho^{\frac{1}{2}(\gamma-1)} - \rho_0^{\frac{1}{2}(\gamma-1)}\}, \quad a^2 = \gamma\rho^{\gamma-1}. \tag{3.3}$$

Here ρ_0, p_0 and α_0 are the values of the state variables on the leading characteristic and without loss of generality the units of pressure have been chosen so that $p_0/\rho_0^\gamma = 1$. The relations (3.3) hold at any fixed station x on any wavelet $\phi = \text{const}$.

4. Variation of wave strength in propagation

Since (3.3) gives the relations between u, ρ, p and α, then on substituting (3.2) into (2.23) this may be reduced to

$$\frac{\partial u}{\partial x} + \tfrac{1}{2}a\left\{\left(\frac{\partial H}{\partial \alpha}\right)\Big/\left(\rho\frac{\partial H}{\partial \rho}\right)\right\}f(p, \rho, \alpha)/(u + a) = 0, \tag{4.1}$$

where, from (3.3) and (2.8)

$$(u + a) = \frac{2\gamma^{\frac{1}{2}}}{\gamma - 1}(\tfrac{1}{2}(\gamma + 1)\rho^{\frac{1}{2}(\gamma-1)} - \rho_0^{\frac{1}{2}(\gamma-1)}). \tag{4.2}$$

Along with (4.1) and (4.2) we have

$$\partial T/\partial x = (u+a)^{-1}. \tag{4.3}$$

Equations (4.1), (4.2) and (4.3) along with the remaining boundary conditions $\mathbf{u}(x^*, \alpha)$, where $x^* = x^*(t)$ is known and may be a piston path, determine the solution.

In the above the notation of §2, where capital letters are used to denote functional dependence on the variables (x, ϕ), has been dropped. But it should be remembered that the solution is being obtained in terms of these variables and that the dependence of ϕ on (x, t) is obtained through (4.3).

As no particular form for $f(p, \rho, \alpha)$ has been considered the simplification of linearization is now introduced by taking

$$\rho = \rho_0 + \rho', \quad a = a_0 + a', \quad u = u', \quad \text{etc.}, \tag{4.4}$$

where the primed quantities are small perturbations of the equilibrium values. In this instance (4.1), (4.2) and (4.3) reduce to

$$\frac{\partial u'}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial H}{\partial \alpha}\right)_0 \left\{\left(\frac{\partial f}{\partial p}\right)_0 a_0^2 + \left(\frac{\partial f}{\partial \rho}\right)_0\right\} \Big/ a_0 \left(\frac{\partial H}{\partial \rho}\right)_0\right] u' = 0 \tag{4.5}$$

and

$$\frac{\partial T}{\partial x} = a_0^{-1}\left\{1 - \frac{\gamma+1}{2a_0} u'\right\}, \tag{4.6}$$

and it has been assumed that $f(p, \rho, \alpha)$ may be expanded in terms of its arguments at equilibrium. The terms which have been neglected are $O\{(\rho'/\rho_0)^2\}$ and $O\{\rho'(\partial\rho'/\partial x)/\rho_0^2\}$. Since the latter term involves derivatives with respect to the variable x in the co-ordinate system (x, ϕ), the above is in no way equivalent to the usual linear theories (see Lick 1967).

Equation (4.5) integrates to give

$$u' = g(\phi) \exp[-\lambda(x - x^*)], \quad (x \geqslant x^*), \tag{4.7}$$

where

$$\lambda = \frac{1}{2}\left[\left(\frac{\partial H}{\partial \alpha}\right)_0 \left\{\left(\frac{\partial f}{\partial p}\right)_0 a_0^2 + \left(\frac{\partial f}{\partial \rho}\right)_0\right\} \Big/ a_0 \left(\frac{\partial H}{\partial \rho}\right)_0\right],$$

and $g(\phi) = u'(x^*, t)$, i.e. ϕ is the time that a 'wavelet' leaves the station x^*. It is expected that λ will be positive. On substituting (4.7) in (4.6) and integrating, the equation of any 'wavelet', $\phi =$ const., is obtained:

$$a_0(T - \phi) = (x - x^*) + \frac{\gamma+1}{2a_0\lambda} g(\phi)(\exp[-\lambda(x - x^*)] - 1). \tag{4.8}$$

The formation of a shock wave is characterized by $\partial T/\partial \phi = 0$ so that by (4.8), if $g'(\phi) > 0$, the prime denoting differentiation with respect to ϕ, then such will occur where

$$\frac{\partial T}{\partial \phi} = 1 + \frac{(\gamma+1) g'(\phi)}{2a_0^2\lambda}(\exp[-\lambda(x - x^*)] - 1) = 0. \tag{4.9}$$

In particular a shock will occur on the wave-front at that value of x obtained from (4.9) by setting $\phi = 0$. The acceleration on any characteristic or wavelet is obtained from (4.7) as

$$\frac{\partial u'}{\partial t} = g'(\phi)\left(\frac{\partial T}{\partial \phi}\right)^{-1} \exp[-\lambda(x - x^*)]. \tag{4.10}$$

Along the leading wavelet or wave-front the strength of the discontinuity in acceleration is obtained through (4.9) and (4.10) by the relation

$$\left.\frac{\partial u'}{\partial t}\right|_{\phi=0} = g'(0)\exp\left[-\lambda(x-x^*)\right]\left\{1+\frac{\gamma+1}{2a_0^2\lambda}g'(0)\left(\exp\left[-\lambda(x-x^*)\right]-1\right)\right\}^{-1}. \tag{4.11}$$

These results for the behaviour of acceleration discontinuities on a wave-front propagating into an equilibrium state have been obtained by Coleman & Gurtin (1967*b*).

The relatively undistorted approximation is valid if

$$|g'(\phi)/g(\phi)\,a_0\lambda| \gg \left(1+\frac{\gamma+1}{2a_0}g(\phi)\exp\left[-\lambda(x-x^*)\right]\right)\frac{\partial T}{\partial\phi}, \tag{4.12}$$

which is satisfied automatically at a wave-front $\phi = 0$ where $g(0) = 0$ or near a shock where $\partial T/\partial\phi = 0$. It is also satisfied in the degenerate case of $(a_0\lambda)\to 0$ in which case the results for an ideal classical non-dissipative gas are obtained in the limit.

At $x = x^* = 0$, $\partial T/\partial\phi = 1$ and the approximation is valid if the local frequency

$$\omega_L = |g'(\phi)/g(\phi)| \gg (\partial T/\partial\phi)\,a_0\lambda = a_0\lambda. \tag{4.13}$$

The validity of the approximation may be extended to all values of (x, ϕ) provided

$$|g'(\phi)/a_0^2\lambda| < M, \quad |g(\phi)/a_0| \ll 1, \tag{4.14}$$

where M is finite, and (4.13) is satisfied. The conditions (4.14) may be satisfied by small-amplitude high-frequency sound waves, i.e. the frequency is high in a sense relative to the natural time $(a_0\lambda)^{-1}$. The relation (4.13) suggests a parameter for an asymptotic analysis. We note through (4.9) that time periodic disturbances at $x = 0$ will not remain periodic in x for $x > 0$. In particular, shocks may form.

A linearized theory would yield instead of (4.11) the result

$$\left.\frac{\partial u'}{\partial t}\right|_{\phi=0} = g'(0)\exp\left[-\lambda(x-x^*)\right]$$

through the assumption

$$|g'(\phi)/g(\phi)\,a_0\lambda| = O(1),$$

which we have seen to be completely erroneous in the above instances.

5. Weak shock waves

By a simple variant of the treatment of shock waves given by Serrin (1959) it may be shown that the behaviour of a dissipative gas through a shock is exactly similar to that of its non-dissipative counterpart. In particular the relations

$$[\alpha] = 0, \quad [\eta] \geqslant 0,$$

where the brackets denote the discontinuity in a variable across the shock, must hold. For weak shocks the entropy jump is third order in the density jump, while the shock speed U is that of the local speed of sound to a first approximation, i.e. $U \simeq a$.

In the limit of weak shocks the relations (3.3) appropriately linearized satisfy the compatibility conditions which must hold across a shock, i.e. the jumps in any variable when computed from (3.3) for two values of ϕ satisfy these conditions.

Since two characteristics, say ϕ_1 and ϕ_2, coalesce at a shock it follows from (4.7) that

$$[u] = [g(\phi_1) - g(\phi_2)] \exp[-\lambda(x - x^*)]. \tag{5.1}$$

The speed G of the shock surface is then given by

$$G = \tfrac{1}{2}\{(a_1 + u_1) + (a_2 + u_2)\} \tag{5.2}$$

to a first approximation and through (4.3) the relation

$$\frac{1}{G} \simeq a_0^{-1}\left(1 - \frac{\gamma+1}{4a_0}\{g(\phi_1) + g(\phi_2)\} \exp[-\lambda(x - x^*)]\right) \tag{5.3}$$

is then derived.

Also at the shock $t_1 = t_2$ and $x_1 = x_2$ where (x_1, t_1) and (x_2, t_2) are the co-ordinates of a point on ϕ_1 and ϕ_2 respectively. Therefore through (4.8) it is implied that at the shock

$$\frac{\phi_1 - \phi_2}{g(\phi_1) - g(\phi_2)} = -\frac{\gamma+1}{2a_0^2\lambda}(\exp[-\lambda(x - x^*)] - 1). \tag{5.4}$$

In general characteristics have the explicit form

$$t = f(x, \phi) + \phi \tag{5.5}$$

and any curve which is intersected by these curves may be represented in (x, ϕ) co-ordinates. Since the shock will be described by a curve $t = s(x)$, it follows from (5.5) and the implicit function theorem that along the shock

$$\phi = \psi(x), \tag{5.6}$$

say. Therefore on the shock wave we have, on substituting (5.6) in (5.5),

$$t = f(x, \psi(x)) + \psi(x).$$

Considering the specific form of (5.5), which is (4.8), we derive a further relation for the shock speed G:

$$\frac{1}{G} = \frac{ds(x)}{dx} \simeq a_0^{-1}\left\{1 - \frac{\gamma+1}{2a_0} g(\phi) \exp[-\lambda(x - x^*)]\right\} + \left\{1 + \frac{\gamma+1}{2a_0^2\lambda} g'(\phi)(\exp[-\lambda(x - x^*)] - 1)\right\}\frac{d\phi}{dx}, \tag{5.7}$$

and this holds for both the ϕ_1 and ϕ_2 sets of characteristics or wavelets. Equations (5.3) and (5.7) then imply that on the shock the relation

$$[g(\phi_1) - g(\phi_2) - (\phi_1 - \phi_2)g'(\phi_1)]\frac{d\phi_1}{dx} = [g(\phi_2) - g(\phi_1) - (\phi_2 - \phi_1)g'(\phi_2)]\frac{d\phi_2}{dx} \tag{5.8}$$

must be satisfied by ϕ_1 and ϕ_2. The shock path is then determined by (4.8), (5.4) and (5.8).

The above derivation parallels that given by Varley & Cumberbatch (1966) for cylindrical and spherical waves in a non-dissipative ideal gas.

In the case of a shock propagating into an undisturbed region then (5.3) and (5.7) yield the relation

$$-\frac{\gamma+1}{4a_0^2}g(\phi_2)\exp[-\lambda(x-x^*)] + \left\{1+\frac{\gamma+1}{2a_0^2\lambda}g'(\phi_2)(\exp[-\lambda(x-x^*)]-1)\right\}\frac{\partial\phi_2}{\partial x} = 0, \quad (5.9)$$

as $\phi_1 = 0$, and this integrates to

$$-\frac{\gamma+1}{2a_0^2\lambda}(\exp[-\lambda(x-x^*)]-1) = 2\int_0^{\phi_2} g(s)ds/g^2(\phi_2). \quad (5.10)$$

This result is similar to that obtained by Whitham (1956), whose result follows from (5.10) in the limit $\lambda \to 0$. Taking the limit of (5.10) as ϕ_2 tends to zero we find

$$\lim_{\phi_2\to 0} 2\int_0^{\phi_2} g(s)ds/g^2(\phi_2) = [g'(0)]^{-1}, \quad (5.11)$$

which confirms the result obtained from (4.9), viz. that the shock first occurs when $\partial T/\partial\phi = 0$. If the compressive phase of a wave is followed by one of rarefaction, then $g(\phi)$ has a zero and some of the wavelets in the neighbourhood of this 'zero' wavelet will not catch up the shock. It follows that the integral in (5.10) is bounded and that at large distance

$$g(\phi_2) \propto \left\{-\frac{\gamma+1}{2a_0^2\lambda}(\exp[-\lambda(x-x^*)]-1)\right\}^{-\frac{1}{2}}. \quad (5.12)$$

From (4.7) and (5.12) it then follows that

$$[u] \propto \left\{\frac{\gamma+1}{2a_0^2\lambda}(1-\exp[-\lambda(x-x^*)])\exp[2\lambda(x-x^*)]\right\}^{-\frac{1}{2}}. \quad (5.13)$$

Similarly, the distance by which the shock is ahead of the 'zero' wavelet $t = \phi_0 + x/\eta_0$, increases by an amount

$$\ell \propto \left\{\frac{\gamma+1}{2a_0^2\lambda}(1-\exp[-\lambda(x-x^*)])\right\}^{\frac{1}{2}}. \quad (5.14)$$

In the limit $\lambda \to 0$, all of the above reduce to those obtained by Whitham (1956) for a non-dissipative ideal gas.

6. Asymptotic analysis

The analysis in §4 has suggested a parameter $a_0\lambda$ with which to form an asymptotic analysis. It was seen there that the 'undistorted' approximation was valid provided (4.13) and (4.14) were satisfied. In this section the propagation of 'high-frequency' harmonic waves is considered. At $x = 0$, the initial conditions are taken to be

$$x = \omega^{-2}\sigma(1-\cos\beta), \quad u = \omega^{-1}\sigma\sin\beta, \quad t = \omega^{-1}\beta \quad (6.1)$$

so that a given wavelet is described by β = const. and the conditions (4.13) and (4.14) are seen to be satisfied for $\omega/a_0\lambda \gg 1$. The constant σ is then the maximum acceleration and is finite.

Again, the transformation of § 2 is employed so that

$$t = T(x, \beta : \omega). \tag{6.2}$$

Equations (6.1) and (6.2), imply the further boundary condition

$$\partial T/\partial \beta = \omega^{-1} - \omega^{-2}\sigma(a+u)^{-1}\sin\beta. \tag{6.3}$$

In terms of characteristic co-ordinates the equations to be satisfied are

$$\left.\begin{aligned} \left(B_{ij}\frac{\partial T}{\partial x} - A_{ij}\right)\frac{\partial u_j}{\partial \beta} &= \frac{\partial T}{\partial \beta}\left(B_{ij}\frac{\partial u_j}{\partial x} + C_i\right), \\ \frac{\partial T}{\partial x} &= (a+u)^{-1}. \end{aligned}\right\} \tag{6.4}$$

We now consider asymptotic expansions of the form

$$\left.\begin{aligned} &u = \sum_{n=1}^{N} \epsilon^n u_n(x,\beta), \quad p = p_0 + \sum_{n=1}^{N} \epsilon^n p_n(x,\beta), \quad \rho = \rho_0 + \sum_{n=1}^{N} \epsilon^n \rho_n(x,\beta), \\ &a = a_0 + \sum_{n=1}^{N} \epsilon^n a_n(x,\beta), \quad \alpha = \alpha_0 + \sum_{n=1}^{N} \epsilon^n \alpha_n(x,\beta), \quad f = \sum_{n=1}^{N} \epsilon^n f_n(x,\beta), \\ &T = T_0(x) + \sum_{n=1}^{N} \epsilon^n T_n(x,\beta), \end{aligned}\right\} \tag{6.5}$$

as these are suggested by the conditions (6.1) and (6.3). The constants appearing in (6.5) are the equilibrium values of the respective variables and $\epsilon = \omega^{-1}$, while successive terms such as u_n and u_{n+1} have the ratio $u_n/u_{n+1} = O(a_0\lambda)$.

By equation (6.4) $\partial \mathbf{u}/\partial \beta$ is not uniquely determined in terms of $\partial \mathbf{u}/\partial x$ and $\mathbf{C}$ on characteristic curves described by the eigenvalue $\partial T/\partial x$ and by (6.5) this has an asymptotic expansion. There exists for each eigenvalue a left eigenvector $\boldsymbol{l}$ and since there is an asymptotic expansion for each eigenvalue it is implied that a similar expansion exists for $\boldsymbol{l}$, viz.

$$\boldsymbol{l} = \boldsymbol{l}_0 + \sum_{n=1}^{N} \epsilon^n \boldsymbol{l}_n. \tag{6.6}$$

Also, on any characteristic curve the relations

$$\boldsymbol{l}\left(\mathbf{B}\frac{\partial T}{\partial x} - \mathbf{A}\right)\frac{\partial \mathbf{u}}{\partial \beta} = \boldsymbol{l}\left(\mathbf{B}\frac{\partial \mathbf{u}}{\partial x} + \mathbf{C}\right) = 0 \tag{6.7}$$

must hold.

Equations (6.4), (6.5) and (6.7) form the basis of the approximating scheme.

Zeroth approximation

On substituting (6.5) into (6.4) and equating coefficients of zero powers of ϵ to zero in both the resulting relation and (6.7), we obtain

$$\left.\begin{aligned} \frac{\partial T_0}{\partial x} &= \frac{1}{a_0}, \\ \boldsymbol{l}_0\left\{\mathbf{B}_0\frac{\partial T_0}{\partial x} - \mathbf{A}_0\right\} &= 0. \end{aligned}\right\} \tag{6.8}$$

The solution of these appropriate to the boundary condition (6.2) are

$$\left.\begin{aligned} T_0 &= x/a_0 \\ \text{and} \qquad \boldsymbol{l}_0 &= \left\{\rho_0\left(\frac{\partial H}{\partial \rho}\right)_0,\quad a_0\left(\frac{\partial H}{\partial \rho}\right)_0,\quad -a_0,\quad a_0\left(\frac{\partial H}{\partial \alpha}\right)_0\right\}.\end{aligned}\right\} \tag{6.9}$$

First approximation

Similarly it is found, by equating coefficients of the first power of ϵ to zero, that

$$\left.\begin{aligned} \frac{\partial T_1}{\partial x} &= -(a_1+u_1)a_0^{-2}, \\ \left(B_{ij}^{(0)}\frac{\partial T_0}{\partial x}-A_{ij}^{(0)}\right)\frac{\partial u_j^{(1)}}{\partial \beta} &= 0, \\ l_i^{(0)}\left(B_{ij}^{(0)}\frac{\partial u_j^{(1)}}{\partial x}+C_i^{(1)}\right) &= 0, \\ l_i^{(0)}\left\{B_{ij}^{(0)}\frac{\partial T_1}{\partial x}+B_{ij}^{(1)}\frac{\partial T_0}{\partial x}-A_{ij}^{(1)}\right\}+l_i^{(1)}\left\{B_{ij}^{(0)}\frac{\partial T_0}{\partial x}-A_{ij}^{(1)}\right\} &= 0. \end{aligned}\right\} \tag{6.10}$$

The first three of equations (6.10) and the appropriate boundary conditions from (6.1) and (6.2), viz. $u_1 = \sigma\sin\beta$, $T_1 = \beta$, have the solution

$$\left.\begin{aligned} u_1 &= \sigma\sin\beta\exp\left[-\lambda(x-x^*)\right], \\ a_0(T_1-\beta) &= \frac{\gamma+1}{2a_0\lambda}\sigma\sin\beta\left(\exp\left[-\lambda(x-x^*)\right]-1\right). \end{aligned}\right\} \tag{6.11}$$

When these, appropriately factored by ω^{-1}, are added to (6.9) then the total is exactly that solution obtained in §4. Also we have the relations that

$$p_1 = a_0\rho_0 u_1, \quad \rho_1 = (\rho_0/a_0)u_1. \tag{6.12}$$

A usual linear theory would replace (6.10) by (6.8) (see Lick 1967).

The solution of the fourth equation of (6.10) for $\boldsymbol{l}^{(1)}$ contains $\boldsymbol{l}^{(0)}$ but this may be subtracted as being superfluous. It may be readily verified that the solution of correct physical dimensions is

$$\boldsymbol{l}^{(1)} = \left[\left\{\rho_1\left(\frac{\partial H}{\partial \rho}\right)_0+\rho_0\left(\frac{\partial H}{\partial \rho}\right)_1\right\},\quad \left\{a_1\left(\frac{\partial H}{\partial \rho}\right)_0+a_0\left(\frac{\partial H}{\partial \rho}\right)_1\right\},\quad -a_1,\quad \left\{a_1\left(\frac{\partial H}{\partial \alpha}\right)_0+a_0\left(\frac{\partial H}{\partial \alpha}\right)_1\right\}\right]. \tag{6.13}$$

By reference to the solution in §4, it is implied by (4.9) that shocks will occur on all wavelets for which $g'(\beta)$ is a maximum, i.e. on $\beta = 2\pi n$ $(n = 0, 1, 2, \ldots)$. These will first be located at the station

$$x = -\frac{1}{\lambda}\log\left(1-\frac{2a_0^2\lambda}{(\gamma+1)\sigma}\right). \tag{6.14}$$

Also through (5.3) it is seen that the shocks formed on $\beta = 2\pi n$ $(n = 1, 2, \ldots)$ propagate with the speed of sound and so a constant distance apart. The leading shock however moves ahead of that on $\beta = 2\pi$ as is indicated by (5.14).

Any two wavelets $\beta_2 = 2\pi n + \phi$ and $\beta_1 = 2\pi n - \phi$ with $n \neq 0$ coalesce with the shock at the same instant and the substitution

$$g(\beta_2) = -g(\beta_1) = \omega^{-1}\sigma \sin\phi \tag{6.15}$$

satisfies (5.7). By (5.4) these two wavelets reach the shock where

$$\frac{\gamma+1}{2a_0^2\lambda}(\exp[-\lambda(x-x^*)]-1) = -\frac{\phi}{\sigma\sin\phi}. \tag{6.16}$$

Therefore it is implied by (6.16) and (5.1) that the strength of these shocks ($\beta = 2\pi n, n = 1, 2, \ldots$) decays like

$$[u] \propto \exp[-\lambda(x-x^*)]/(1-\exp[-\lambda(x-x^*)]).$$

Those wavelets β which lie in the region

$$(2n-2)\pi+\phi < \beta < (2n)\pi - \phi \quad (n = 1, 2, \ldots)$$

never coalesce with a shock and so form the expansion regions separating the shocks.

Second approximation

The analysis required to obtain the full second approximation is algebraically complicated but as it is of interest to enquire into the behaviour of the variable α, which has remained constant up to the first approximation, this will now be sketched out. The second-order equations are

$$\partial T_2/\partial x = \{(a_1+u_1)^2/a_0 - (a_2+u_2)\}a_0^{-2}, \tag{6.17a}$$

$$\left\{B_{ij}^{(0)}\frac{\partial T_0}{\partial x} - A_{ij}^{(0)}\right\}\frac{\partial u_j^{(2)}}{\partial \beta} = -\left\{B_{ij}^{(0)}\frac{\partial T_1}{\partial x} + B_{ij}^{(1)}\frac{\partial T_0}{\partial x} - A_{ij}^{(1)}\right\}\frac{\partial U_j^{(1)}}{\partial \beta} + \frac{\partial T_1}{\partial \beta}\left(B_{ij}^{(0)}\frac{\partial u_j^{(1)}}{\partial x} + C_i^{(1)}\right), \tag{6.17b}$$

$$l_i^{(0)}\left(B_{ij}^{(0)}\frac{\partial u_j^{(2)}}{\partial x} + C_i^{(2)}\right) = -l_i^{(0)}B_{ij}^{(1)}\frac{\partial u_j^{(1)}}{\partial x} - l_i^{(1)}\left(B_{ij}^{(0)}\frac{\partial u_j^{(1)}}{\partial x} + C_i^{(1)}\right), \tag{6.17c}$$

while the remaining equation for $\boldsymbol{l}^{(2)}$ is not considered.

In (6.17*b*) the rank of the matrix on the left is three and so we choose to ignore the third of the system of equations which (6.17*b*) represents. Employing (6.11) and (6.12) and integrating

$$-u_2 + (a_0\rho_0)^{-1}p_2 = \int_0^\beta \left[-a_0\lambda\left\{1+\frac{\gamma+1}{2a_0^2\lambda}g'(r)(\exp[-\lambda s]-1)\right\} + \frac{\gamma+1}{2a_0}g'(r)\exp[-\lambda s]\right]g(r)\exp[-\lambda s]\,dr, \tag{6.18a}$$

$$\frac{\rho_0}{a_0}u_2 - \rho_2 = \int_0^\beta \left[-\rho_0\lambda\left\{1+\frac{\gamma+1}{2a_0^2\lambda}g'(r)(e^{-\lambda s}-1)\right\} + \frac{(\gamma-3)\rho_0}{2a_0^2}g'(r)e^{-\lambda s}\right]g(r)e^{-\lambda s}dr, \tag{6.18b}$$

$$\alpha_2 = \int_0^\beta \frac{\rho_0}{a_0}\left\{1+\frac{\gamma+1}{2a_0^2\lambda}g'(r)(e^{-\lambda s}-1)\right\}f_1(s,r)\,dr, \tag{6.18c}$$

where

$$g(\beta) = \sigma\sin\beta, \quad f_1(s,\beta) = \left\{\left(\frac{\partial f}{\partial p}\right)_0 a_0^2 + \left(\frac{\partial f}{\partial \rho}\right)_0\right\}g(\beta)e^{-\lambda s}, \quad s = (x-x^*).$$

The arbitrary functions of x which arise in the integration are zero as the functions u_2, p_2, ρ_2 and α_2 are zero on $\beta = 0$ (the leading characteristic) for all x. The last of these, viz. α_2, is independent of the others but is dependent in its behaviour on what has occurred on the precursor wavelets. Thus, on integrating (6.18*c*),

$$\alpha_2 = \frac{\rho_0}{a_0}\left\{\left(\frac{\partial f}{\partial p}\right)_0 a_0^2 + \left(\frac{\partial f}{\partial \rho}\right)_0\right\} e^{-\lambda s}\left\{\frac{\gamma+1}{4a_0^2\lambda} g^2(\beta)(e^{-\lambda s}-1) + \int_0^\beta g(r)\,dr\right\}, \quad (6.19)$$

and it is seen that at the 'piston', $x = x^*$, the degree of internal excitation induced is

$$\alpha_2 \propto \int_0^\beta g(r)\,dr, \quad (6.20)$$

which is out of phase with the velocity wave. Its period is twice that of velocity and its effect is to induce frequency 'dispersion' in the velocity wave.

As in the first approximation the variation of u_2, p_2 and ρ_2 is obtained by substituting (6.18) into (6.17*c*). With the particular forms of the specific enthalpy given by (2.10) and (2.15) the zeroth and first-order coefficients in (6.6), which are given by (6.8) and (6.13), are

$$\left.\begin{aligned} l^{(0)} &= \left\{-a_0^2/(\gamma-1),\quad a_0^3/\rho_0(\gamma-1),\quad a_0,\quad -a_0\left(\frac{a_0^2\delta}{\gamma-1}+d\right)\right\},\\ l^{(1)} &= \left\{-a_0u_1,\quad \frac{a_0^2(5-3\gamma)}{2\rho_0(\gamma-1)}U_1,\quad -\frac{\gamma-1}{2}u_1,\quad (\tfrac{3}{2}a_0^2\delta+d)\,u_1\right\},\end{aligned}\right\} \quad (6.21)$$

where

$$\delta = \frac{\partial}{\partial\alpha}\left\{\log\left(\frac{C_{p,\alpha}}{1+\alpha}\right)\right\}\Bigg|_{\alpha=\alpha_0},$$

which is of course zero if the enthalpy has the form (2.15). After some algebra the solution for u_2 satisfying (6.1) and (6.3) is found to be

$$\begin{aligned} u_2 = &-\frac{A}{\lambda}e^{-\lambda s}\left\{\frac{\gamma+1}{4a_0^2\lambda}g^2(\beta)(e^{-\lambda s}+\lambda s-1) - \lambda s\int_0^\beta g(r)\,dr\right\}\\ &-Bg^2(\beta)\,e^{-\lambda s}(e^{-\lambda s}-1) - \frac{a_0\lambda}{2}\left\{\frac{\gamma+1}{4a_0^2\lambda}g^2(\beta) - \int_0^\beta g(r)\,dr\right\}s\,e^{-\lambda s}, \end{aligned} \quad (6.22)$$

where

$$A = -\frac{1}{2}\left(\delta+\frac{(\gamma-1)d}{a_0^2}\right)\left[\rho_0\lambda\left\{\left(\frac{\partial f}{\partial \rho}\right)_0 - a_0^2\left(\frac{\partial f}{\partial p}\right)_0\right\} - \frac{2a_0^2\lambda}{a_0^2\delta+(\gamma-1)d}\left(\frac{\partial f}{\partial \alpha}\right)_0\right],$$

$$B = \rho_0\left\{\frac{\gamma+1}{4}\left(\frac{\partial f}{\partial p}\right)_0 - \frac{\gamma-3}{4a_0^2}\left(\frac{\partial f}{\partial \rho}\right)_0\right\} + \left(\delta+\frac{(\gamma-1)d}{a_0^2}\frac{\rho_0^2}{4a_0^2}\right)\left\{C - \frac{7\lambda}{2} - \frac{2a_0^2\delta}{a_0^2\delta+d(\gamma-1)}\right\},$$

$$C = \left(\frac{\partial^2 f}{\partial \rho^2}\right)_0 + 2a_0^2\left(\frac{\partial^2 f}{\partial p\,\partial \rho}\right)_0 + a_0^4\left(\frac{\partial^2 f}{\partial p^2}\right)_0.$$

The integral term in (6.22) is less rapid in its decay than any other term so that at large distance its effect is greater.

On substituting (6.18) and (6.22) into (6.17), the second correction to the characteristic relation satisfying (6.1) and (6.3) is obtained in the form

$$\begin{aligned} T_2 = \frac{\sigma\cos\beta}{a_0} - \frac{(\gamma+1)A}{2(a_0\lambda)^2}\bigg[\frac{\gamma+1}{8a_0^2\lambda}g^2(\beta)\{(e^{-2\lambda s}-1)+2\lambda\, e^{-\lambda s}\} \\ -\{\lambda s\, e^{-\lambda s}+(e^{-\lambda s}-1)\}\int_0^\beta g(r)\,dr\bigg] \\ -\frac{\gamma+1}{4a_0^2\lambda}Bg^2(\beta)(e^{-\lambda s}-1)(e^{-\lambda s}+3)-\frac{\gamma+1}{4a_0\lambda}\bigg\{\frac{\gamma+1}{4a_0^2\lambda}g^2(\beta) \\ -\int_0^\beta g(r)\,dr\bigg\}\{(\lambda s+1)\,e^{-\lambda s}-1\} \\ -\frac{D}{\lambda}\bigg\{\frac{\gamma+1}{4a_0^2\lambda}g^2(\beta)(e^{-\lambda s}+1)-\int_0^\beta g(r)\,dr\bigg\}(e^{-\lambda s}-1) \\ -\frac{3\gamma(\gamma-2)+1}{2\lambda}g^2(\beta)(e^{-\lambda s}-1), \end{aligned} \tag{6.23}$$

where
$$D = \frac{a_0^3\lambda\delta(C_{v,\alpha}/R_2)_0^{-2}}{(\gamma-1)(a_0^2\delta+(\gamma-1)d)} - \tfrac{1}{2}\{a_0\lambda(\gamma+1)\},$$

and again $\delta = 0$ if the form (2.15) is chosen for the enthalpy. This relation (6.23) shows that the disturbance maintains its full cycle period $2\pi/\omega$ but that due to the dependence of T_2 on $\int_0^\beta g(r)\,dr$ frequency 'dispersion' is present.

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